B.Sc. Physics Part-III

Mathematical Physics.
(Paper-V, Group A)

Cauchy-Riemann Conditions:-

Let us consider a function $f(z)$, where $z = x+iy$.
Next, we use partial differential rule to obtain $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial z}\cdot 1 = \frac{\partial f}{\partial z} \\ \text{and } \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial f}{\partial z}\cdot i \end{aligned}\right\} \quad (1)$$

$$\left\{\because z = x+iy,\ \frac{\partial z}{\partial x} = 1 \text{ and } \frac{\partial z}{\partial y} = i\right\}$$

Further, we have the relation $f = u(x,y) + iv(x,y)$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \text{ and } \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \quad (2)$$

Now using relation (1) and (2), we obtain

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad (3)$$

$$\text{and } \frac{\partial f}{\partial z} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad (4)$$

Now since $\cancel{\frac{\partial f}{\partial z}}$. $\frac{df}{dz}$ exits and is unique, the definition of analytic function. this implies that $\frac{\partial f}{\partial z}$ in (3) and (4) must be equal. Thus, we write

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ ;\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \leftarrow \text{Cauchy-Riemann conditions-}$$